Solving systems of linear simultaneous equations

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1 Introduction

There are several methods of solving systems of linear equations. Some of these methods are more efficient than others, some require more memorization, and some require more mathematical background. The purpose of this document is to list out a few which are very handy, along with how they work, how you can implement them, and a few tips for their application. I also provide you with some further reading.

2 Methods

Here we list the methods and steps.

2.1 The Method of Substitution (i.e. Brute force)

Most people learned this one in high school math. If you forget everything else, this is likely to be the one you'll recall in order to be able to solve a system of equations.

Basically, given a system of equations (usually two if you are doing this by hand, but it could be three or more - it gets inconveniently complicated after three), with n-equations and n-unknowns, you can solve those equations by first solving in turn for one variable, then substituting that equivalent calculation for the variable in a second or other equation. In turn you will eliminate the additional variables in the final equation until you are left with one equation which has a single unknown. You solve for that unknown then substitute back into the other equations, solving in a stepwise fashion each equation, until you have all the variables computed.

In fact, this is essentially what you do with linear algebra, and the whole idea there is that there is a convenient and organized way to compute these values with a minimum number of steps and computations. Plus by realizing that these things are all related by various rules, you can exploit that to implement the solution algorithms on a computer and save yourself a huge amount of time. One can solve systems with thousands upon thousands of variables with a computer. You would take your entire lifetime solving that by hand, while a computer can do it in a fraction of a second (using tools such as Matlab - see [2]). Yay!

2.1.1 Steps

1. Rearrange all equations so the variables are on the left, and the constants are on the right of the equal sign (It may help to keep things neat by organizing variables so they are vertically matching, and line up the equal signs)
2. Choose whichever equation seems simpler, and solve for the first variable in terms of the others
3. Substitute that result into the next equation
4. Continue that process until you have an equation with a single unknown, solve that equation to get a value
5. Substitute that, in reverse order, back up the chain and solve for the next variable back (in the case of a 2-variable system, you first solve for x, then use that to solve for y, then substitute y's answer back into the first equation with both x and y, and solve for x)

2.1.2 Example

Given the following pair of equations, solve for x and y.

$$\begin{aligned} x + 5y &= 2 \\ -2x + 21y &= 10 \end{aligned} \tag{1}$$

We choose to solve for x in the top equation out of convenience:

$$x = 2 - 5y \tag{2}$$

Now we substitute back into the second equation to find y:

$$-2(2 - 5y) + 21y = 10 \tag{3}$$

We simplify,

$$-4 + 10y + 21y = 10 \tag{4}$$

which gives

$$31y = 14 \tag{5}$$

or

$$y = \frac{14}{31} \tag{6}$$

We will leave it as a fraction for now. Now we take y and substitute that value back into the first equation to solve for x,

$$x + 5\left(\frac{14}{31}\right) = 2 \tag{7}$$

which gives

$$x = 2 - 5\left(\frac{14}{31}\right) \tag{8}$$

which is

$$x = \frac{62 - 70}{31} = -\frac{8}{31} \tag{9}$$

So the answer is

$$x = -\frac{8}{31}, \tag{10}$$

$$y = \frac{14}{31} \tag{11}$$

and just to be sure we substitute that into the second equation to check that it works:

$$-2(-\frac{8}{31}) + 21(\frac{14}{31}) = 10? \tag{12}$$

well let's compute the left side and see if it matches the right side of the equal sign.

$$\frac{16}{31} + \frac{294}{31} = \frac{310}{31} = 10 \tag{13}$$

So indeed this was correct.

2.2 Cramer's Rule (i.e. fast - put me into your note page!)

Cramer's rule is a formula used in linear algebra which can allow the user to solve a system of n linear equations and n unknowns if the system has a unique solution (see [1]). The solution is 'valid' whenever a solution exists for the system. What does this mean? Well, basically sometimes a system of linear equations does not have an exact solution, and instead one can compute the closest solution. We will not discuss existence and uniqueness of solutions here in detail, the reader is referred to [4] or another linear algebra text for further information. Basically, if we have a linear equation $Ax = b$, and we want to know if we can invert the matrix A in order to solve for x, we can quickly test if A is invertible by computing the determinant of A. If the determinant is zero, the matrix is singular, or non-invertible, and other methods must be used to compute the closest approximation to the inverse and an answer for x. Given the matrix A,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{14}$$

recall that to compute the determinant of a 2x2 matrix,

$$|A| = ad - bc \tag{15}$$

so this is fairly easy to test.

But we get ahead of ourselves. Let us examine the steps to compute the solution of a system of linear equations using Cramer's Rule for the following system (of n linear equations with n unknowns),

$$Ax = b \tag{16}$$

where A has a non-zero determinant, and x is given by $x = \{x_1, x_2...x_n\}^T$ (and T denotes the transpose of a matrix or vector), and $b = \{b_1, b_2, ...b_n\}^T$ is a vector of contants

2.2.1 Steps

1. Form the matrices A_i by replacing the ith column of A by the column vector b
2. Then compute the answer for each x_i as $x_i = \frac{|A_i|}{|A|}$, where $|\cdot|$ represents the determinant operation

2.2.2 Example

For any set of two equations with two unknowns,

$$\begin{aligned} a_1x_1 + a_2x_2 &= b_1 \\ a_3x_1 + a_4x_2 &= b_2 \end{aligned} \tag{17}$$

a solution by Cramer's Rule works in the following way,

$$x_1 = \frac{\begin{vmatrix} b_1 & a_2 \\ b_2 & a_4 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}} = \frac{b_1a_4 - a_2b_2}{a_1a_4 - a_2a_3} \tag{18}$$

$$x_2 = \frac{\begin{vmatrix} a_1 & b_1 \\ a_3 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}} = \frac{a_1b_2 - b_1a_3}{a_1a_4 - a_2a_3} \tag{19}$$

Try solving via substitution the above equations, and simplify. Does the answer come out to be the same, in terms of the coefficients?

2.3 Linear Algebra - other methods

If you have access to a calculator that can perform matrix operations (or a numerical package such as Matlab, Octave, or R), then you can simply arrange equations in standard form as above. Then define the matrix A and vectors x and b. Compute the inverse of A and left-multiply b by A^{-1}.

There are a number of ways to compute this manually, such as via Gaussian elimination (see [4]), but we will only mention that here.

There are many subtleties to efficient application of these methods in order to minimize processor time. So this becomes important for huge numbers of variables, it is less important for only a few. But the interested reader is pointed to [3] or one of the many other excellent texts out there.

2.3.1 Steps

1. Arrange equations in standard form - organize the variables on the left, with the variables in the same place for all equations (i.e. if you have variables x_1 and x_2, the equation might be arranged as

$$a_1x_1 + a_2x_2 = b \tag{20}$$

2. Define the A coefficient matrix, the variable vector x, and the constant vector, b
3. Determine if the inverse of A exists by computing the determinant, or using another test (see [4])
4. Compute A^{-1}
5. Left-multiply both sides by the inverse of A to get

$$x = A^{-1}b \tag{21}$$

2.3.2 Example - Matlab

Consider the equations given by the same ones as in Section 2.1.2.

Matlab has many useful functions for computing these values. It can also directly solve systems of linear equations as well, but here we will discuss a few individual steps.

We will input the matrix A by typing

```
A = [1, 5; -2, 21];
```

and

```
b = [2; 10];
```

Compute the determinant of A to test for singularity by typing

```
det(A)
```

which we do, and we find that the determinant of A is 31, so it is non-singular. So next we type

```
x = inv(A) * b
```

which gives the output

$$x = \tag{22}$$
$$-0.2581 \tag{23}$$
$$0.4516$$
$$\tag{24}$$

which is the decimal equivalent to four significant figures of the fractional answer from Section 2.1.2.

3 Practice problems

Determine x_1 and x_2 for the following using all the methods outlined above, and confirm that the answers match. Time yourself for each method and note the time it takes.

3.1 Problem 1

$$x_1 - 2x_2 = 10 \tag{25}$$
$$x_2 = 5x_1 - 30$$

3.2 Problem 2

$$4x_1 + 10x_2 = 1 \tag{26}$$
$$2x_1 = 2x_2 + 2$$

3.3 Problem 3

$$\begin{aligned} 30x_1 + x_2 &= -1 \\ x_1 + 3x_2 &= -2 \end{aligned} \tag{27}$$

4 Tips

4.1 Tips for clarification of confusing topics

Matrix manipulation is not only a mathematical methodology but can also be an art form. A master of these techniques can use matrix manipulation to determine characteristics of the physics of a system without solving the entire problem, can determine the controllability of a feedback control system as well as the dynamical behavior of the open loop system. There are many applications for linear algebra and matrix methods, so it is a great tool for a student in the physical or other sciences to learn.

If you find your original course in linear algebra was confusing, try dropping by a professor's office who teaches it, and ask a few questions. They will be happy you have interest to learn it. If your book was unclear to you, there are many excellent books out there, and the internet is a great resource.

Practice problems. It is a very important aspect to these techniques. Memorize rules, and review the book from time to time to refresh in your memory. Read papers that have been published on interesting topics that use these techniques.

Sit in on courses about the topics you may be unclear on.

4.2 Cramer's rule

Don't forget to check for singularity. It is rare that a random set of equations will be singular, especially for 2x2 systems, but it can happen. Indeed, in control theory, a singular matrix can come about from many potential physical properties or configurations of systems.

Don't just memorize the output but rather the method of Cramer's Rule. That way you are less likely to make a small error since you can check the equation you think you remember as for how it goes.

4.3 Linear algebra

There are many computational tools out there these days that can perform matrix manipulation. Not only commercial ones like Matlab, but open source libraries for any programming language you use. Also, if you have a graphing calculator, it probably can do matrix manipulation, so take some time to learn it.

In Matlab, if you compute the inverse of A using the inv() command, it will pretty much always give you an answer, even if the matrix is singular, so keep that in mind. Always check whether the matrix can be inverted before attempting to solve the problem.

5 Conclusion

Here we have reviewed a few basic techniques for computing the solution of linear systems of equations. This is not a complete set of the possible strategies that can be used, but it is useful, and can provide a review for those needing practice.

References

[1] Cramer's rule, wikipedia.

[2] The mathworks page, http://www.mathworks.com.

[3] J. Ferziger. *Numerical Methods for Engineering Application.* John Wiley and Sons, New York, NY, 2nd edition edition, 1998.

[4] David C. Lay. *Linear Algebra and Its Applications.* Addison Wesley, 2nd edition edition, 1996.