*12-3 DISCONTINUITY FUNCTIONS

The method of integration, used to find the equation of the elastic curve for a beam or shaft, is convenient if the load or internal moment can be expressed as a *continuous function* throughout the beam's *entire length*. If several different loadings act on the beam, however, the method becomes more tedious to apply, because separate loading or moment functions must be written for each region of the beam. Furthermore, integration of these functions requires the evaluation of integration constants using boundary conditions and/or continuity conditions. For example, the beam shown in Fig. 12–14 requires four moment functions to be written. They describe the moment in regions AB, BC, CD, and DE. When applying the moment-curvature relationship, $EI\,d^2v/dx^2 = M$, and integrating each moment equation twice, we must evaluate *eight* constants of integration. These involve *two* boundary conditions that require zero displacement at points A and E, and *six* continuity conditions for both slope and displacement at points B, C, and D.

In this section we will discuss a method for finding the equation of the elastic curve for a *multiply loaded beam* using a *single expression*, either formulated from the loading on the beam, $w = w(x)$, or the beam's internal moment, $M = M(x)$. If the expression for w is substituted into $EI\,d^4v/dx^4 = -w(x)$ and integrated four times, or if the expression for M is substituted into $EI\,d^2v/dx^2 = M(x)$, and integrated twice, the constants of integration will be determined only from the boundary conditions. Since the continuity equations will not be involved, the analysis will be greatly simplified.

Discontinuity Functions. In order to express the load on the beam or the internal moment within it using a single expression, we will use two types of mathematical operators known as *discontinuity functions*.

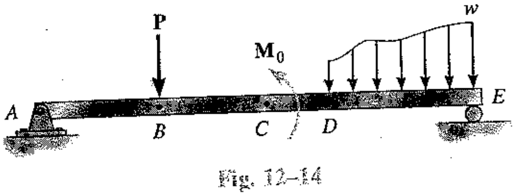


Fig. 12–14

FROM: HIBBELER, "MECHANICS OF MATERIALS"
PRENTICE HALL

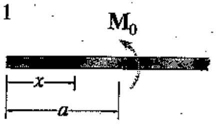
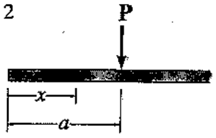
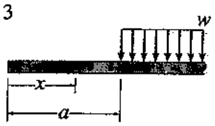
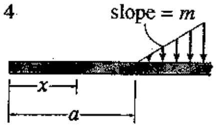
Loading	Loading Function $w=w(x)$	Shear $V=-\int w(x)dx$	Moment $M=\int Vdx$
1 	$w = M_0\langle x-a\rangle^{-2}$	$V = -M_0\langle x-a\rangle^{-1}$	$M = -M_0\langle x-a\rangle^{0}$
2 	$w = P\langle x-a\rangle^{-1}$	$V = -P\langle x-a\rangle^{0}$	$M = -P\langle x-a\rangle$
3 	$w = w_0\langle x-a\rangle^{0}$	$V = -w_0\langle x-a\rangle^{1}$	$M = \frac{-w_0}{2}\langle x-a\rangle^{2}$
4 	$w = m\langle x-a\rangle^{2}$	$V = \frac{-m}{2}\langle x-a\rangle^{2}$	$M = \frac{-m}{6}\langle x-a\rangle^{3}$

Table 12–2

Macaulay Functions. For purposes of beam or shaft deflection, Macaulay functions, named after the mathematician W. H. Macaulay, are used only to describe ***distributed loadings***. They can be written in general form as

$$\boxed{\langle x-a\rangle^n = \begin{cases} 0 & \text{for } x < a \\ (x-a)^n & \text{for } x \geq a \end{cases} \quad n \geq 0} \tag{12–11}$$

Here x represents the coordinate position of a point along the beam and a is the location on the beam where a "discontinuity" occurs, namely the point where a distributed loading *begins*. Note that the Macaulay function $\langle x-a\rangle^n$ is written with angle brackets to distinguish it from the ordinary function $(x-a)^n$, written with parentheses. As stated by the equation, only when $x \geq a$ is $\langle x-a\rangle^n = (x-a)^n$, otherwise it is zero. Furthermore, these functions are valid only for exponential values $n \geq 0$. Integration of Macaulay functions follows the same rules as for ordinary functions, i.e.,

$$\int \langle x-a\rangle^n\,dx = \frac{\langle x-a\rangle^{n+1}}{n+1} + C \tag{12–12}$$

Note how the Macaulay functions describe both the *uniform load* w_0 $(n = 0)$ and *triangular load* $(n = 1)$, shown in Table 12–2, items 3 and 4. This type of description can, of course, be extended to distributed loadings having other forms. Also, it is possible to use superposition with the uniform and triangular loadings to create the Macaulay function for a trapezoidal loading. Using integration, the Macaulay functions for shear, $V = -\int w(x)\,dx$, and moment, $M = \int V\,dx$, are also shown in the table.

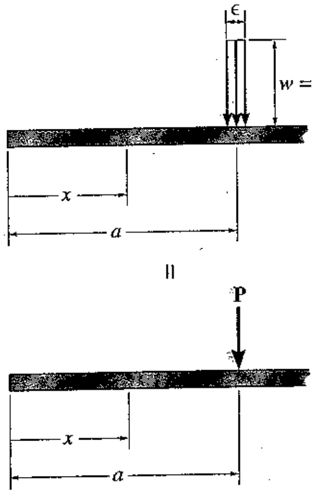


Fig. 12–15

Singularity Functions. These functions are used only to describe the point location of concentrated forces or couple moments acting on a beam or shaft. Specifically, a ***concentrated force* P** can be considered as a special case of a distributed loading, where the intensity of the loading is $w = P/\epsilon$ and its width is ϵ, where $\epsilon \rightarrow 0$, Fig. 12–15. The area under this loading diagram is equivalent to P, *positive downward*, and so we will use the singularity function

$$w = P\langle x - a\rangle^{-1} = \begin{cases} 0 & \text{for } x \neq a \\ P & \text{for } x = a \end{cases} \tag{12–13}$$

to describe the force P. Note that here $n = -1$ so that the units for w are force per length, as it should be. Furthermore, the function takes on the value of P only at the point $x = a$ where the load occurs, otherwise it is zero.

In a similar manner, a couple moment $\mathbf{M}_0$, considered *positive counterclockwise*, is a limitation as $\epsilon \rightarrow 0$ of two distributed loadings as shown in Fig. 12–16. Here the following function describes its value.

$$w = M_0\langle x - a\rangle^{-2} = \begin{cases} 0 & \text{for } x \neq a \\ M_0 & \text{for } x = a \end{cases} \tag{12–14}$$

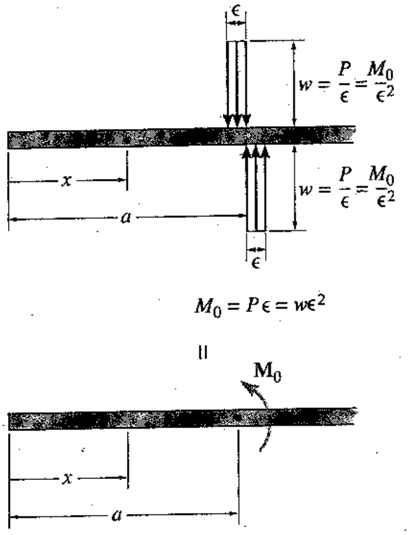


Fig. 12–16

The exponent $n = -2$, in order to ensure that the units of w, force per length, are maintained.

Integration of the above two singularity functions follows the rules of operational calculus and yields results that are *different* from those of Macaulay functions. Specifically,

$$\int \langle x - a\rangle^{n} = \langle x - a\rangle^{n+1},\ n = -1, -2 \tag{12–15}$$

Here, only the exponent n increases by one, and no constant of integration will be associated with this operation. Using this formula, notice how M_0 and P, described in Table 12–2, items 1 and 2, are integrated once, then twice, to obtain the internal shear and moment in the beam.

Application of Eqs. 12–11 through 12–15 provides a rather direct means for expressing the loading or the internal moment in a beam as a function of x. When doing so, close attention must be paid to the signs of the external loadings. As stated above, and as shown in Table 12–2, *concentrated forces and distributed loads are positive downward, and couple moments are positive counterclockwise.* If this sign convention is followed, then the internal shear and moment are in accordance with the beam sign convention established in Sec. 6.1.

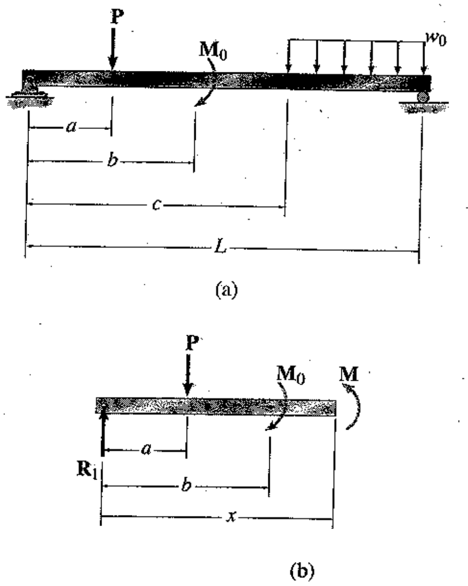


Fig. 12–17

As an example of how to apply discontinuity functions to describe the loading or internal moment in a beam, we will consider the beam loaded as shown in Fig. 12–17*a*. Here the reactive force $\mathbf{R}_1$ created by the pin, Fig. 12–17*b*, is negative since it acts upward, and $\mathbf{M}_0$ is negative since it acts clockwise. Using Table 12–2, the loading at any point x on the beam is therefore,

$$w = -R_1\langle x - 0\rangle^{-1} + P\langle x - a\rangle^{-1} - M_0\langle x - a\rangle^{-2} + w_0\langle x - c\rangle^{0}$$

The reactive force at the roller is not included here since x is never greater than L, and furthermore, this value is of no consequence in computing slope or deflection. Note that when $x = a$, $w = P$, all other terms being zero. Also, when $x > c$, $w = w_0$, etc.

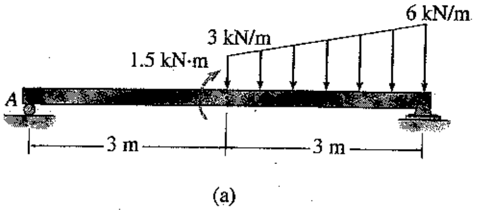
Integrating this equation twice yields the expression that describes the internal moment in the beam. The constants of integration will be ignored here since the boundary conditions, or the end shear and moment, have been calculated ($V = R_1$ and $M = 0$) and these values are incorporated into the beam loading w. One can also obtain this result directly from Table 12–2. In either case,

$$M = R_1\langle x - 0\rangle - P\langle x - a\rangle + M_0\langle x - b\rangle^{0} - \tfrac{1}{2}w_0\langle x - c\rangle^{2} \qquad (12\text{–}16)$$

The validity of this expression may be checked by using the method of sections, say, within the region $b < x < c$, Fig. 12–17*b*. Moment equilibrium requires that

$$M = R_1 x - P(x - a) + M_0 \qquad (12\text{–}17)$$

This result agrees with that obtained from the discontinuity functions, since by Eqs. 12–13 and 12–15 only the last term in Eq. 12–16 is zero when $x < c$.



(a)

As a second example, consider the beam in Fig. 12–18*a*. The support reaction at A has been computed in Fig. 12–18*b*, and the trapezoidal loading has been separated into triangular and uniform loadings. From Table 12–2, the loading is therefore

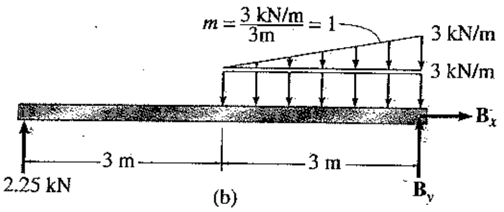
$$w = -2.25\ \text{kN}\langle x - 0\rangle^{-1} - 1.5\ \text{kN}\cdot\text{m}\langle x - 3\ \text{m}\rangle^{-2} + 3\ \text{kN/m}\langle x - 3\ \text{m}\rangle^{0} + 1\langle x - 3\ \text{m}\rangle^{2}$$

We can determine the moment expression directly from Table 12–2, rather than integrating this expression twice. In either case,

$$M = 2.25\ \text{kN}\langle x - 0\rangle^{1} + 1.5\ \text{kN}\cdot\text{m}\langle x - 3\ \text{m}\rangle^{0} - \frac{3\ \text{kN/m}}{2}\langle x - 3\ \text{m}\rangle^{2} - \frac{1}{6}\langle x - 3\ \text{m}\rangle^{3}$$

$$= 2.25x - 1.5 - 1.5\langle x - 3\rangle^{2} - \frac{1}{6}\langle x - 3\rangle^{3}$$

The slope and deflection of the beam can now be determined after this equation is integrated two successive times and the constants of integration are evaluated using the boundary conditions of zero displacement at A and B.



(b)

Fig. 12–18

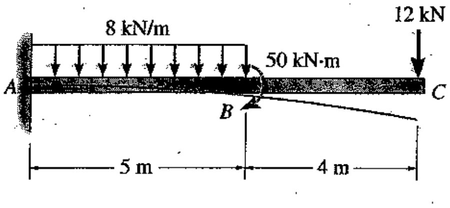
PROCEDURE FOR ANALYSIS

The following procedure provides a method for using discontinuity functions to determine a beam's elastic curve. This method is particularly advantageous for solving problems involving beams or shafts subjected to *several loadings*, since the constants of integration can be evaluated by using *only* the boundary conditions, while the compatibility conditions are automatically satisfied.

Elastic Curve. Sketch the beam's elastic curve and identify the boundary conditions at the supports. Recall that zero displacement occurs at all pin and roller supports, and zero slope and zero displacement occur at fixed supports. Establish the x axis so that it extends to the right and has its origin at the beam's left end.

Loading or Moment Function. Calculate the support reactions and then use the discontinuity functions in Table 12–2 to express either the loading w or the internal moment M as a function of x. When doing so, make sure to follow the sign convention for each loading as it applies for this equation. Also, note that the distributed loadings must extend all the way to the beam's right end to be valid. If this does not occur, use the method of superposition, which is illustrated in Example 12–5.

Slope and Elastic Curve. Substitute w into $EI\,d^4v/dx^4 = -w(x)$ or M into the moment-curvature relation $EI\,d^2v/dx^2 = M$, and integrate to obtain the equations for the beam's slope and deflection. Evaluate the constants of integration using the boundary conditions and substitute these constants into the slope and deflection equations to obtain the final results. When these functions are evaluated at any point on the beam, a *positive slope* is *counterclockwise*, and a *positive displacement* is *upward*.



(a)

Fig. 12–19

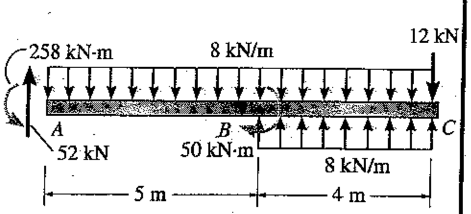
EXAMPLE 12–5

Determine the equation of the elastic curve for the cantilevered beam shown in Fig. 12–19*a*. EI is constant.

SOLUTION

Elastic Curve. The loads cause the beam to deflect as shown in Fig. 12–19*a*. The boundary conditions require zero slope and displacement at A.

Loading Function. The support reactions at A have been calculated by statics and are shown on the free-body diagram in Fig. 12–19*b*. Since the distributed loading in Fig. 12–19*a* does not extend to C as required, we can use the superposition of loadings shown in Fig. 12–19*b* to represent the same effect. By our sign convention, the 50-kN · m couple moment, the 52-kN force at A, and the portion of distributed loading from B to C on the bottom of the beam are all negative. With reference to Fig. 12–19*b*, the beam's loading is therefore



(b)

$$w = -52\text{ kN}\langle x - 0\rangle^{-1} + 258\text{ kN}\cdot\text{m}\langle x - 0\rangle^{-2} + 8\text{ kN/m}\langle x - 0\rangle^{0} - 50\text{ kN}\cdot\text{m}\langle x - 5\text{ m}\rangle^{-2} - 8\text{ kN/m}\langle x - 5\text{ m}\rangle^{0}$$

The 12-kN load is *not included* here, since x cannot be greater than 9 m. Because $dV/dx = -w(x)$, then by integrating, neglecting the constant of integration since the reactions are included in the load function, we have

$$V = 52\langle x - 0\rangle^{0} - 258\langle x - 0\rangle^{-1} - 8\langle x - 0\rangle^{1} + 50\langle x - 5\rangle^{-1} + 8\langle x - 5\rangle^{1}$$

Furthermore, $dM/dx = V$, so that integrating again yields

$$M = -258\langle x - 0\rangle^{0} - (-52)\langle x - 0\rangle - \frac{1}{2}(8)\langle x - 0\rangle^{2} - \frac{1}{2}(-8)\langle x - 5\rangle^{2} - (-50)\langle x - 5\rangle^{0}$$

$$= (-258 + 52x - 4x^{2} + 4\langle x - 5\rangle^{2} + 50\langle x - 5\rangle^{0})\text{ kN}\cdot\text{m}$$

This same result can be obtained *directly* from Table 12–2.

Slope and Elastic Curve. Applying Eq. 12–10 and integrating twice, we have

$$EI\frac{d^{2}v}{dx^{2}} = -258 + 52x - 4x^{2} + 4\langle x - 5\rangle^{2} + 50\langle x - 5\rangle^{0}$$

$$EI\frac{dv}{dx} = -258x + 26x^{2} - \frac{4}{3}x^{3} + \frac{4}{3}\langle x - 5\rangle^{3} + 50\langle x - 5\rangle^{1} + C_1$$

$$EIv = -129x^{2} + \frac{26}{3}x^{3} - \frac{1}{3}x^{4} + \frac{1}{3}\langle x - 5\rangle^{4} + 25\langle x - 5\rangle^{2} + C_1x + C_2$$

Since $dv/dx = 0$ at $x = 0$, $C_1 = 0$; and $v = 0$ at $x = 0$, so $C_2 = 0$. Thus,

$$v = \frac{1}{EI}\left(-129x^{2} + \frac{26}{3}x^{3} - \frac{1}{3}x^{4} + \frac{1}{3}\langle x - 5\rangle^{4} + 25\langle x - 5\rangle^{2}\right)\text{ m} \qquad \textit{Ans.}$$

Note: If we *did not* use discontinuity functions, two x coordinates would be needed for regions AB and BC, and the problem would require evaluating two more constants of integration involving the continuity of slope and displacement at point B.